

On the Union-Closed Sets Conjecture

ANAGHA G .

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1 Introduction

The Union-Closed Sets Conjecture is easy to state but remains open. A family of sets is *union-closed* if the union of any two member sets is again a member of the family.

Let $[m] = \{1, 2, \dots, m\}$ and let $\mathcal{F} \subseteq 2^{[m]}$ be finite. The family $\mathcal{F}$ is union-closed if

$$A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}.$$

Union-Closed Sets Conjecture

If $\mathcal{F}$ is a finite union-closed family containing a nonempty member set, then some element of its universe belongs to at least half of the member sets of $\mathcal{F}$.

The constant $1/2$ is best possible. For example, in the full power set $2^{[m]}$, every element belongs to exactly 2^{m-1} of the 2^m member sets.

2 Preliminaries

First, we fix some notation.

The *universe* of $\mathcal{F}$ is

$$U(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} F.$$

For $x \in U(\mathcal{F})$, define

$$\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}, \quad \mathcal{F}_{\bar{x}} = \mathcal{F} \setminus \mathcal{F}_x.$$

The number $|\mathcal{F}_x|$ is the *frequency* of x . If $\mathcal{F}$ is union-closed, then both $\mathcal{F}_x$ and $\mathcal{F}_{\bar{x}}$ are union-closed.

Intersection Formulation

There are several equivalent formulations of the Union-Closed Conjecture, one of which is the Intersection-Closed Conjecture.

Let $\mathcal{F}$ be union-closed and put $U = U(\mathcal{F})$. After adjoining $\emptyset$ if necessary, taking complements in U gives

$$\mathcal{D} = \{U \setminus F : F \in \mathcal{F}\}.$$

By De Morgan's law, $\mathcal{D}$ is *intersection-closed*: if $A, B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$.

Intersection-Closed Conjecture

If $\mathcal{D}$ is a finite intersection-closed family with at least two distinct member sets, then some element of its universe belongs to at most half of its member sets.

An element $x \in U(\mathcal{F})$ is *abundant* if $|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|$. In the intersection-closed formulation, an element $y \in U(\mathcal{D})$ is *rare* if $|\mathcal{D}_y| \leq \frac{1}{2}|\mathcal{D}|$. Complementation shows that the two formulations are equivalent.

We use the following standard conventions and reductions:

1. The family and all its member sets are finite, so its universe is finite.
2. The family contains a nonempty member set; the exceptional family $\{\emptyset\}$ has no element to test.
3. We may adjoin $\emptyset$ to a union-closed family without destroying union-closure or making the conjecture easier.

Definition. A union-closed family $\mathcal{F}$ is *separating* if, for every two distinct elements $x, y \in U(\mathcal{F})$, some member set contains exactly one of x and y .

3 Examples

For $S = \{1, 2, 3, 4\}$, the power set 2^S is union-closed and has 16 member sets. Each element of S occurs in exactly 8 of them, so equality holds in the conjectured bound.

A smaller non-Boolean example is

$$\mathcal{F} = \{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}.$$

This family is union-closed. The frequencies of 1, 2, 3 are respectively 2, 3, 2, so all three elements are abundant; 1 and 3 meet the threshold exactly.

There is also a genuine group-theoretic consequence of the lattice formulation discussed below. Abdollahi, Woodroffe, and Zaimi proved that the subgroup lattice of every finite group satisfies Frankl's lattice conjecture [22]. Since the join-irreducibles of a subgroup lattice are precisely the cyclic subgroups of prime-power order, every nontrivial finite group has an element of prime-power order that is contained in at most half of its subgroups.

4 A Brief History

The conjecture is generally attributed to Péter Frankl, who formulated it in 1979, although it has also been described as folklore [20]. The first published references appeared in the 1980s. A later Polymath project investigated strengthenings, variants, and possible counterexamples. The conjecture itself remains open.

5 Where's the Bottleneck?

The difficulty is that union-closure supplies little information about where an abundant element should occur. Bruhn and Schaudt group many approaches into three broad methods: injections, local configurations, and averaging [20].

5.1 Injection

Suppose that $\mathcal{F}$ is union-closed and $\{x\} \in \mathcal{F}$. Then

$$\phi : \mathcal{F}_{\bar{x}} \longrightarrow \mathcal{F}_x, \quad A \longmapsto A \cup \{x\},$$

is well-defined and injective. Therefore

$$|\mathcal{F}_{\bar{x}}| \leq |\mathcal{F}_x| \implies 2|\mathcal{F}_x| \geq |\mathcal{F}_x| + |\mathcal{F}_{\bar{x}}| = |\mathcal{F}|,$$

so x is abundant.

The same broad idea shows that if a union-closed family contains a 2-element member, at least one of those two elements is abundant. The pattern fails for 3-element members: there are union-closed families with a unique smallest member of size 3 such that none of its elements is abundant [20].

5.2 Special Cases

The local-configuration method asks for a small union-closed family $\mathcal{L}$ with the following property: every union-closed family containing $\mathcal{L}$ has an abundant element in $U(\mathcal{L})$. Such a family is called *Frankl-complete*, or *FC*. Poonen gave an exact characterization of FC-families by a finite system of nonnegative weighting inequalities [6]. This converts some special cases into finite linear-feasibility problems and has been used both in computational verification and in proofs for structured graph classes.

5.3 Averaging

Write $n = |\mathcal{F}|$ and $m = |U(\mathcal{F})|$. Double-counting incidences gives

$$\sum_{x \in U(\mathcal{F})} |\mathcal{F}_x| = \sum_{F \in \mathcal{F}} |F|.$$

Consequently, if the average set size is at least $m/2$, then the average element frequency is at least $n/2$, and some element is abundant. This observation is the basis of the averaging method.

Theorem (Nishimura and Takahashi [1]). Let $|U(\mathcal{F})| = m$. If $\mathcal{F}$ is union-closed and has more than

$$2^m - \frac{1}{2}\sqrt{2^m}$$

member sets, then $\mathcal{F}$ satisfies the Union-Closed Sets Conjecture.

Proof. Suppose that $S \subseteq U(\mathcal{F})$, $|S| \geq m/2$, but $S \notin \mathcal{F}$. For each pair $\{R, S \setminus R\}$ of complementary subsets of S , at most one member can belong to $\mathcal{F}$, since otherwise union-closure would give $S \in \mathcal{F}$. Thus at least

$$2^{|S|-1} \geq \frac{1}{2}\sqrt{2^m}$$

subsets of the universe are absent from $\mathcal{F}$, contradicting the assumed size. Hence every subset of size at least $m/2$ belongs to $\mathcal{F}$. Pair each member of size less than $m/2$ with its complement, which is also a member and has size greater than $m/2$. The unpaired members have size at least $m/2$, so the average set size is at least $m/2$. The incidence identity above then gives an abundant element. $\square$

Theorem (Reimer [2]). If $\mathcal{F}$ is union-closed and $n = |\mathcal{F}|$, then

$$\frac{1}{n} \sum_{F \in \mathcal{F}} |F| \geq \frac{1}{2} \log_2 n.$$

Combining Reimer's theorem with the incidence identity shows that some element has frequency at least

$$\frac{n \log_2 n}{2m},$$

or equivalently appears in at least a $\frac{\log_2 n}{2m}$ fraction of the member sets. This is generally below the conjectured fraction $1/2$.

Theorem (Falgas-Ravry [3]). Let $\mathcal{F}$ be a separating union-closed family on m elements. If $|\mathcal{F}| \leq 2m$, then $\mathcal{F}$ satisfies the Union-Closed Sets Conjecture.

Proof. Label the elements $1, \dots, m$ in nondecreasing order of frequency. For each $i < j$, separation and the frequency ordering imply that some $X_{ij} \in \mathcal{F}$ contains j but not i : otherwise $\mathcal{F}_j$ would be a proper subset of $\mathcal{F}_i$. Put $X_0 = U(\mathcal{F})$ and

$$X_i = \bigcup_{j=i+1}^m X_{ij} \quad (1 \leq i < m).$$

For $1 \leq i < m$, the set X_i contains every $j > i$ but omits i ; consequently $X_0, X_1, \dots, X_{m-1}$ are distinct. All of them contain the most frequent element m . Hence m has frequency at least m , which is at least $|\mathcal{F}|/2$ under the hypothesis. $\square$

6 Classes for which the Conjecture holds

Let $n = |\mathcal{F}|$ and $m = |U(\mathcal{F})|$. Some established special cases are:

- The conjecture holds when $m \leq 12$, and consequently when $n \leq 50$ [7, 8].
- It holds for separating families with $n \leq 2m$ [3].
- Balla, Bollobás, and Eccles proved it when $n \geq \frac{2}{3} 2^m$ [4].
- Karpas proved that there is an absolute constant $c > 0$ such that it holds whenever $n \geq (\frac{1}{2} - c) 2^m$ [24].

Theorem (Knill [9]). Any union-closed family with $n \geq 2$ member sets has an element of frequency at least

$$\frac{n-1}{\log_2 n}.$$

Proof. Adjoin $\emptyset$ if necessary, and let the resulting family have $N \in \{n, n+1\}$ member sets. Choose $S \subseteq U(\mathcal{F})$ minimal subject to meeting every nonempty member set. For each $x \in S$, minimality gives an $A_x \in \mathcal{F}$ with $A_x \cap S = \{x\}$. Taking unions of the sets A_x shows that every subset of S occurs as the trace on S of a member set, and hence $N \geq 2^{|S|}$.

The $N - 1$ nonempty member sets all meet S , so some $x \in S$ has frequency at least

$$\frac{N - 1}{|S|} \geq \frac{N - 1}{\log_2 N}.$$

If $N = n$ this is the claimed bound. If $N = n + 1$, the bound is $n/\log_2(n + 1) \geq (n - 1)/\log_2 n$ for $n \geq 2$. Since adjoining $\emptyset$ changes no element frequency, the same element works in the original family. $\square$

Wójcik slightly improved Knill's estimate [10]. In asymptotic notation, these results give an element of frequency $\Omega(n/\log n)$. They have since been superseded for large n by the constant-proportion bounds discussed in Section 10.

We have also managed to make use of bounds on the size of the universe itself.

Theorem (Lo Faro [8]). Suppose the conjecture fails, and let

$$m_0 = \min\{|U(\mathcal{A})| : \mathcal{A} \text{ is a counterexample}\}.$$

Then every counterexample has at least $4m_0 - 1$ member sets.

7 Graph Formulation

An independent set of a graph G is a set of pairwise nonadjacent vertices. It is *maximal* if no further vertex can be added while preserving independence. Bruhn, Charbit, Schaudt, and Telle proved that the following graph conjecture is equivalent to the Union-Closed Sets Conjecture [12].

Every bipartite graph with at least one edge has, in each bipartition class, a vertex that lies in at most half of its maximal independent sets.

The equivalence passes through the intersection-closed formulation: traces of maximal independent sets on one bipartition class form an intersection-closed family.

The graph conjecture is known for several natural classes. In particular, it holds for chordal bipartite graphs, subcubic bipartite graphs, bipartite series-parallel graphs, and bipartitioned circular-interval graphs [12]. Here *chordal bipartite* means that the graph has no induced cycle of length greater than 4, while *subcubic* means that every vertex has degree at most 3.

8 Lattice Formulation

The conjecture can also be phrased in terms of finite lattices. A poset $(L, \leq)$ is a lattice if every pair $x, y \in L$ has a greatest lower bound $x \wedge y$ (the meet) and a least upper

bound $x \vee y$ (the join). The power set of a finite set, ordered by inclusion, is a lattice whose meet is intersection and whose join is union.

An element j of a finite lattice is *join-irreducible* if it is not the least element and $j = a \vee b$ implies $j = a$ or $j = b$. Equivalently, j cannot be expressed as a join of strictly smaller elements. Write $[j, \hat{1}] = \{x \in L : j \leq x\}$.

Every finite lattice L with at least two elements has a join-irreducible element j such that

$$|[j, \hat{1}]| \leq \frac{1}{2}|L|.$$

This lattice statement is equivalent to the intersection-closed, and hence the union-closed, formulation [20]. Poonen proved it for geometric and complemented lattices [6]; Abe and Nakano proved it for modular lattices [18]; and Reinhold extended the result to lower semimodular lattices [19]. Abdollahi, Woodroffe, and Zaimi later proved it for the subgroup lattice of every finite group [22].

9 Salzborn Formulation

This equivalent formulation, attributed to Salzborn, concerns a special class of union-closed families called normalized families.

For a finite union-closed family, a *basis set* is a nonempty member that is not the union of other, strictly smaller members. A separating family on m elements has at least m nonempty member sets; if it also contains $\emptyset$, it therefore has at least $m + 1$ member sets.

Definition. A union-closed family $\mathcal{N}$ is *normalized* if $\emptyset \in \mathcal{N}$, the family is separating, and

$$|U(\mathcal{N})| = |\mathcal{N}| - 1.$$

Conjecture (Salzborn [11]). Every normalized family $\mathcal{N} \neq \{\emptyset\}$ contains a basis set B with

$$|B| \geq \frac{1}{2}|\mathcal{N}|.$$

To see one direction of the equivalence, start with a union-closed family $\mathcal{A}$ containing $\emptyset$ and define

$$\mathcal{A}_{\not\subseteq X} = \{A \in \mathcal{A} : A \not\subseteq X\}, \quad \mathcal{A}^* = \{\mathcal{A}_{\not\subseteq X} : X \in \mathcal{A}\}.$$

Then $\mathcal{A}^*$ is normalized, $|\mathcal{A}^*| = |\mathcal{A}|$, and every basis set of $\mathcal{A}^*$ is of the form $\mathcal{A}_x$ for some $x \in U(\mathcal{A})$. Salzborn's bound for a basis set of $\mathcal{A}^*$ therefore gives an abundant element of $\mathcal{A}$. The reverse implication is also known; thus the Salzborn formulation is equivalent to the original conjecture [10, 11, 20].

10 Recent Progress

Gilmer introduced an information-theoretic method and obtained the first positive absolute lower bound, improving the earlier $\Omega(1/\log n)$ bounds.

Theorem (Gilmer [15]). Every nontrivial finite union-closed family has an element contained in at least a 0.01 fraction of its member sets.

Proof. Write

$$h(p) = -p \log_2 p - (1-p) \log_2 (1-p)$$

for binary entropy. We first record Gilmer's one-bit estimate. Suppose that, conditional on a finite random variable $C = c$, a bit X is Bernoulli with parameter p_c , and let (C', X') be an independent copy. If $\mathbb{E}p_C \leq 0.01$, then

$$H(X \vee X' \mid C, C') = \mathbb{E}_{C, C'} h(p_C + p_{C'} - p_C p_{C'}) \geq 1.26 \mathbb{E}_C h(p_C). \quad (1)$$

Here is a short verification. Split the values of C into $C_0 = \{c : p_c \leq 0.1\}$ and $C_1 = C_0^c$. Markov's inequality gives $q_0 := \Pr(C \in C_0) \geq 0.9$. The elementary entropy estimates

$$h(p + p' - pp') \geq 1.4 \frac{h(p) + h(p')}{2} \quad (p, p' \leq 0.1)$$

and

$$h(p + p' - pp') \geq (1-p)h(p') \quad (0 \leq p, p' \leq 1)$$

follow respectively from concavity of h together with $h(0.9s)/h(0.5s) > 1.4$ for $0 < s \leq 0.2$, and directly from concavity applied to $p \cdot 1 + (1-p)p'$. Averaging the first estimate over $C_0 \times C_0$ gives at least $1.4q_0^2 \mathbb{E}[h(p_C) \mid C_0] \geq 1.26q_0 \mathbb{E}[h(p_C) \mid C_0]$. Averaging the second over the two mixed blocks gives at least $1.8q_0q_1 \mathbb{E}[h(p_C) \mid C_1] \geq 1.62q_1 \mathbb{E}[h(p_C) \mid C_1]$. The contribution from $C_1 \times C_1$ is nonnegative, proving (1).

Now choose A and B independently and uniformly from a union-closed family $\mathcal{F}$, and identify them with their indicator vectors. Suppose for a contradiction that every element occurs in fewer than $0.01|\mathcal{F}|$ member sets. For each coordinate i , apply (1) with $C = A_{<i}$ and $C' = B_{<i}$. Since $(A \cup B)_{<i}$ is a function of $(A_{<i}, B_{<i})$, conditioning and the chain rule give

$$\begin{aligned} H(A \cup B) &= \sum_i H((A \cup B)_i \mid (A \cup B)_{<i}) \\ &\geq \sum_i H((A \cup B)_i \mid A_{<i}, B_{<i}) \\ &\geq 1.26 \sum_i H(A_i \mid A_{<i}) = 1.26H(A). \end{aligned}$$

If $|\mathcal{F}| \geq 2$, then $H(A) = \log_2 |\mathcal{F}| > 0$. On the other hand, union-closure implies that $A \cup B$ is supported on $\mathcal{F}$, so

$$H(A \cup B) \leq \log_2 |\mathcal{F}| = H(A),$$

a contradiction. A one-member nontrivial family satisfies the theorem immediately. $\square$

Several authors quickly sharpened the entropy argument. Sawin proved the bound

$$\frac{3 - \sqrt{5}}{2} \approx 0.38197$$

and sketched a refinement giving a strictly larger constant [13]. Yu made that refinement finite-dimensional and obtained the explicit unconditional bound 0.38234 [23]. Chase and Lovett proved that $(3 - \sqrt{5})/2$ is the sharp threshold for an approximate union-closed model [14].

11 Where to from here?

The conjecture remains open. The information-theoretic method now supplies strong universal constants, while FC-family methods, averaging, computation, and the graph and lattice formulations continue to resolve structured cases. The subgroup-lattice case is fully settled [22], so further progress must address lattices and set systems beyond that class. The central difficulty remains converting partial structural or average information into the sharp frequency threshold $1/2$.

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