

A Singular Introduction to Commutative Algebra

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Abstract

These notes introduce commutative rings, ideals, localization, standard bases, and modules, with computations in SINGULAR. They follow Gert-Martin Greuel and Gerhard Pfister's book *A Singular Introduction to Commutative Algebra* [1]. The first part develops monomial orderings and ideal theory; the second extends these ideas to graded modules and their presentations; the final part gives computational examples. Throughout, rings are commutative with identity, ring homomorphisms preserve identity, and modules are unital. We allow the zero ring when explicitly mentioned. We write $\mathbb{N} = \{0, 1, 2, \dots\}$ and use K for a field. Additional background comes from [2]; definitions and software conventions are also checked against [3, 6].

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I Rings, Ideals and Standard Bases

1 Ring Theory

A **ring** R is a set with addition and multiplication such that $(R, +)$ is an abelian group, multiplication is associative, multiplication distributes over addition on both sides, and there is a multiplicative identity 1_R . We work with commutative rings, so $ab = ba$ for $a, b \in R$.

1.1 Basic definitions

- A **unit** is an element $a \in R$ with an inverse $b \in R$, meaning $ab = 1_R$. The units form a group $R^\times$ under multiplication.
- A **field** is a nonzero commutative ring in which every nonzero element is a unit. The nonzero condition excludes the zero ring.
- A **zero divisor** is a nonzero $a \in R$ for which $ab = 0$ for some nonzero $b \in R$.
- An **integral domain** is a nonzero commutative ring with no zero divisors. Equivalently, $ab = 0$ implies $a = 0$ or $b = 0$.

For example, $\mathbb{Z}$ and $K[x_1, \dots, x_n]$ are integral domains. The ring $\mathbb{Z}/6\mathbb{Z}$ has zero divisors, since $\bar{2}\bar{3} = 0$.

2 Monomial Orderings

Write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^n$. A **monomial ordering** $>$ is a strict total order on monomials compatible with multiplication:

$$x^\alpha > x^\beta \implies x^{\alpha+\gamma} > x^{\beta+\gamma} \quad (\gamma \in \mathbb{N}^n).$$

Here we allow global, local, and mixed orderings. Some texts reserve “monomial order” for the global case and include well-ordering in the definition; that convention would exclude the local orders used below.

2.1 Global, local, and mixed orderings

An ordering is **global** if $x_i > 1$ for every variable, **local** if $x_i < 1$ for every variable, and **mixed** otherwise. Equivalently, in the global (respectively local) case every nonconstant monomial is greater (respectively smaller) than 1. A global ordering is a well-order when read from smallest to largest: every nonempty set of monomials has a least element. A local ordering is not a well-order, since $1 > x_i > x_i^2 > \cdots$. Every nonzero polynomial still has a largest monomial, because it has only finitely many terms.

If $f = c_\alpha x^\alpha + \text{smaller terms with } c_\alpha \neq 0$, define

$$\text{LM}(f) = x^\alpha, \quad \text{LC}(f) = c_\alpha, \quad \text{LT}(f) = c_\alpha x^\alpha.$$

We leave $\text{LM}(0)$ undefined. The leading monomial does not include its coefficient.

2.2 Dickson’s lemma

For every $M \subseteq \mathbb{N}^n$ there is a finite $B \subseteq M$ such that for each $\alpha \in M$ some $\beta \in B$ satisfies $\beta_i \leq \alpha_i$ for all i . Thus M has finitely many minimal elements for the componentwise partial order. Equivalently, every monomial ideal in a polynomial ring has a finite monomial generating set.

2.3 Examples

- **Lexicographic order** (lp in SINGULAR): $x^\alpha > x^\beta$ if the first nonzero entry of $\alpha - \beta$ is positive. In particular the first differing exponent satisfies $\alpha_i > \beta_i$; equality does not give a strict comparison.
- **Degree lexicographic order** (Dp): first compare total degree $|\alpha| = \sum_i \alpha_i$, then break ties lexicographically.
- **Degree reverse lexicographic order** (dp): first compare total degree; for equal degree, $x^\alpha > x^\beta$ if the last nonzero entry of $\alpha - \beta$ is negative.

These three orderings are global. Local degree orderings compare smaller total degree as larger; SINGULAR's ds is a local negative-degree reverse lexicographic ordering.

2.4 Weighted and block orderings

For $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$, define

$$\deg_w(x^\alpha) = w \cdot \alpha, \quad \deg_w(f) = \max_{c_\alpha \neq 0} w \cdot \alpha \quad (f \neq 0).$$

A weighted ordering refines this comparison: if $w \cdot \alpha > w \cdot \beta$, then $x^\alpha > x^\beta$; equal weights require a fixed multiplicatively compatible tie-breaker. Positive weights give a global order, and strictly negative weights give a local order. Zero weights require particular care with the tie-breaker. Block orderings compare one group of variables before another; matrix orderings compare exponent vectors after applying a suitable matrix.

3 Ideals and Quotient Rings

3.1 Ideals and generators

An **ideal** I of a ring A is an additive subgroup of A satisfying $aI \subseteq I$ for every $a \in A$. Equivalently, I is nonempty, closed under subtraction, and closed under multiplication by elements of A . For a family $(f_\lambda)_{\lambda \in \Lambda}$ of elements of A ,

$$(f_\lambda \mid \lambda \in \Lambda) = \left\{ \sum_{\lambda \in F} a_\lambda f_\lambda : F \subseteq \Lambda \text{ finite, } a_\lambda \in A \right\}$$

is the ideal it generates. The family generates I if these finite sums are exactly the elements of I . An ideal generated by one element is **principal**; for example, $(n) = n\mathbb{Z}$ in $\mathbb{Z}$.

3.2 Operations on ideals

For ideals $I, J \subseteq A$,

$$I + J = \{a + b : a \in I, b \in J\}, \quad IJ = (ab : a \in I, b \in J), \quad (I : J) = \{a \in A : aJ \subseteq I\}.$$

The intersection $I \cap J$ is also an ideal. Products IJ consist of *finite sums* of products, not just individual products. A union of ideals need not be an ideal, though the union of an ascending chain is. For a ring homomorphism $\phi : A \rightarrow B$, the inverse image of an ideal of B and the kernel of ϕ are ideals of A . The image $\phi(A)$ is a subring of B , but need not be an ideal of B .

3.2.1 Radicals and nilpotents

The **radical** of I is

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some integer } n \geq 1\}.$$

It is an ideal containing I . The ideal I is **radical** if $I = \sqrt{I}$. For an integer $m \geq 1$, let r be the product of the distinct prime divisors of m (with $r = 1$ when $m = 1$). Then $\sqrt{m\mathbb{Z}} = r\mathbb{Z}$; for instance, $\sqrt{8\mathbb{Z}} = 2\mathbb{Z}$. An element a is **nilpotent** if $a^n = 0$ for some $n \geq 1$. The nilpotent elements form the nilradical $\sqrt{(0)}$.

3.3 Noetherian rings

A ring is **Noetherian** if every ideal is finitely generated. The following are equivalent:

1. A is Noetherian.
2. Every ascending chain $I_0 \subseteq I_1 \subseteq \cdots$ stabilizes: there is a j with $I_i = I_j$ for every $i \geq j$.
3. Every nonempty collection of ideals has a maximal element under inclusion.

A field is Noetherian. Hilbert's basis theorem says that $A[x_1, \dots, x_n]$ is Noetherian whenever A is Noetherian.

3.4 Quotient rings

For an ideal $I \subseteq A$, set $a \sim b$ if $a - b \in I$. The cosets $a + I$ form the quotient ring A/I , with

$$(a + I) + (b + I) = (a + b) + I, \quad (a + I)(b + I) = ab + I.$$

The quotient map $\pi : A \rightarrow A/I$, $a \mapsto a + I$, is surjective and has kernel I . Ideals of A/I correspond bijectively to ideals J of A with $I \subseteq J$, by $J \mapsto J/I$ and inverse image under π .

3.5 Prime and maximal ideals

An ideal P is **prime** if $P \neq A$ and $ab \in P$ implies $a \in P$ or $b \in P$. An ideal $\mathfrak{m}$ is **maximal** if $\mathfrak{m} \neq A$ and the only ideals containing it are itself and A . The requirement that these ideals be proper is essential.

- P is prime if and only if A/P is an integral domain.
- $\mathfrak{m}$ is maximal if and only if $A/\mathfrak{m}$ is a field. Hence every maximal ideal is prime.
- If P is prime, $IJ \subseteq P$, and $I \not\subseteq P$, then $J \subseteq P$.
- If $I_1 \cap \cdots \cap I_n \subseteq P$ for a prime P , then $I_i \subseteq P$ for some i ; use $I_1 \cdots I_n \subseteq I_1 \cap \cdots \cap I_n$.
- **Finite prime avoidance:** if $I \subseteq P_1 \cup \cdots \cup P_n$ and every P_i is prime, then $I \subseteq P_i$ for some i .

4 Localization

4.1 Local rings

A **local ring** is a nonzero ring A with exactly one maximal ideal $\mathfrak{m}$. Its residue field is $A/\mathfrak{m}$. A field is local with maximal ideal (0) ; for a prime p and $n \geq 1$, $\mathbb{Z}/p^n\mathbb{Z}$ is local with maximal ideal $(\bar{p})$. Every proper ideal of a nonzero ring is contained in a maximal ideal.

A nonzero ring is local if and only if its *nonunits* form an ideal; that ideal is then its unique maximal ideal. In particular, in a local ring $(A, \mathfrak{m})$ the units are exactly $A \setminus \mathfrak{m}$, and $1 + a$ is a unit for every $a \in \mathfrak{m}$. More generally, the Jacobson radical $J(A)$ consists of those a for which $1 - ab$ is a unit for every $b \in A$; see [3].

4.2 Rings of fractions

A subset $S \subseteq A$ is **multiplicatively closed** if $1 \in S$ and $st \in S$ for $s, t \in S$. One usually excludes 0; allowing it produces the zero localization. We do not assume $S = A \setminus \{0\}$: this set is multiplicatively closed only when A is an integral domain.

The localization $S^{-1}A$ consists of equivalence classes of pairs $(a, s) \in A \times S$, denoted a/s , where

$$(a, s) \sim (b, t) \iff u(at - bs) = 0 \text{ for some } u \in S.$$

Addition and multiplication are given by

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \frac{a}{s} \frac{b}{t} = \frac{ab}{st}.$$

The canonical homomorphism $j : A \rightarrow S^{-1}A$, $a \mapsto a/1$, satisfies $j(a) = 0$ if and only if $sa = 0$ for some $s \in S$. Every $j(s)$ with $s \in S$ is a unit, and $S^{-1}A = 0$ if and only if $0 \in S$. If A is a domain and $S = A \setminus \{0\}$, this construction is the fraction field of A .

4.2.1 Universal property

If $\phi : A \rightarrow B$ sends every element of S to a unit, there is a unique homomorphism $\psi : S^{-1}A \rightarrow B$ with $\psi \circ j = \phi$, given by $\psi(a/s) = \phi(a)\phi(s)^{-1}$. It is an isomorphism if every $b \in B$ has this form and every $a \in \ker \phi$ is killed by some element of S . For a prime ideal $\mathfrak{p}$, the ring $A_{\mathfrak{p}} = (A \setminus \mathfrak{p})^{-1}A$ is local, with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

4.3 Krull dimension

The **Krull dimension** of A is the supremum of the lengths r of strict chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r.$$

A chain with $r + 1$ prime ideals has length r . For a field K , $\dim K[x_1, \dots, x_n] = n$. The chain

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_n)$$

proves the lower bound; the upper bound is the polynomial-ring dimension theorem, not a consequence of this single chain alone.

4.4 Localization associated to an ordering

Set $K[x] = K[x_1, \dots, x_n]$. For a monomial ordering $>$, define

$$S_{>} = \{u \in K[x] \setminus \{0\} : \text{LM}(u) = 1\}, \quad K[x]_{>} = S_{>}^{-1}K[x].$$

Since $\text{LM}(uv) = \text{LM}(u)\text{LM}(v)$, the set $S_{>}$ is multiplicatively closed. Every $u \in S_{>}$ has a nonzero constant term, so it is invertible as a formal power series. For $f/u \neq 0$ in $K[x]_{>}$, define $\text{LM}(f/u) = \text{LM}(f)$ and $\text{LC}(f/u) = \text{LC}(f)/\text{LC}(u)$; these are independent of the chosen representative.

- For a global ordering, $S_{>} = K^\times$ and $K[x]_{>} = K[x]$.
- For a local ordering, $S_{>} = K[x] \setminus (x_1, \dots, x_n)$ and $K[x]_{>} = K[x]_{(x_1, \dots, x_n)}$, the local ring at the origin.
- For any ordering, $K[x] \subseteq K[x]_{>} \subseteq K[x]_{(x_1, \dots, x_n)}$.
- The ring $K[x]_{>}$ is Noetherian and a unique factorization domain, since both properties are preserved by localization of $K[x]$.

4.4.1 Elimination and leading ideals

An ordering on $K[x, y]$ is an **elimination ordering for x** if, for every nonzero polynomial f , $\text{LM}(f) \in K[y]$ implies $f \in K[y]$. A block order that compares the x -block globally before the y -block has this property. It need not be global on the y -variables. For $G \subseteq K[x]_{>}$, its **leading ideal** is the monomial ideal

$$L_{>}(G) = (\text{LM}(g) : g \in G \setminus \{0\}) \subseteq K[x].$$

The polynomial ring in which this ideal is generated must be specified, even though the elements of G may lie in a localization.

5 Normal Forms and Standard Bases

Fix a monomial ordering $>$ and let $R = K[x_1, \dots, x_n]_{>}$. We use the leading monomial and leading ideal defined in Chapter 4.

5.1 Standard bases

For an ideal $I \subseteq R$, a finite set $G \subseteq I \setminus \{0\}$ is a **standard basis** if $L_{>}(G) = L_{>}(I)$. Equivalently, for each nonzero $f \in I$ there is a $g \in G$ with $\text{LM}(g) \mid \text{LM}(f)$. The empty set is a standard basis of the zero ideal. For a global ordering a standard basis is called a **Gröbner basis**. A standard basis generates I as an ideal of R .

A standard basis is **minimal with respect to leading monomials** if no leading monomial divides that of another element. A polynomial is **fully reduced** with respect to G if none of its monomials lies in $L_{>}(G)$. In the global case a **reduced Gröbner basis** is a monic standard basis whose elements are fully reduced with respect to the other basis elements. Minimality of leading monomials alone does not imply that tails are reduced.

For a fixed global ordering every ideal has a unique reduced Gröbner basis; arbitrary Gröbner bases are not unique. For example, both $\{x\}$ and $\{2x\}$ generate $(x) \subseteq \mathbb{Q}[x]$ and are Gröbner bases, but only $\{x\}$ is monic and reduced. Deleting elements with redundant leading monomials from a standard basis gives a minimal one; reducing tails and normalizing coefficients are additional steps. For local or mixed orderings one must distinguish polynomial representatives from their possibly infinite formal expansions; we do not assert the global uniqueness statement in that setting.

5.2 Standard representations and normal forms

Let $G = (g_1, \dots, g_s)$ be a finite list of nonzero elements of R . A representation $h = \sum_i a_i g_i$ is **standard** if, whenever $a_i g_i \neq 0$,

$$\text{LM}(a_i g_i) \leq \text{LM}(h).$$

For $h = 0$ take all $a_i = 0$. This condition prevents terms larger than the leading monomial of h from cancelling in the representation.

A normal-form map assigns to (f, G) an element $r = \text{NF}(f \mid G) \in R$ with the following properties:

1. $\text{NF}(0 \mid G) = 0$.
2. If $r \neq 0$, then $\text{LM}(r) \notin L_{>}(G)$.
3. $f - r$ has a standard representation with respect to G .

The second property concerns the leading monomial only. A *fully reduced* normal form also has no reducible tail monomials; in the global polynomial case it is the usual division remainder. It may depend on

the chosen divisors when G is not a Gröbner basis. With a Gröbner basis and a fixed global ordering, the fully reduced remainder is unique.

5.2.1 Weak normal forms

For local or mixed orderings ordinary polynomial division can fail to terminate. A **weak normal form** retains the first two properties but replaces the third by: there is a unit $u \in R^\times$ such that

$$uf - r = \sum_i a_i g_i$$

is a standard representation. Algorithms such as Mora's normal-form algorithm use this unit multiplier to handle local orderings while working with polynomial representatives; see [1].

If G is a standard basis of I , either notion gives the membership test

$$f \in I \iff \text{NF}(f \mid G) = 0.$$

Indeed, if $f \in I$, then $r \in I$ and a nonzero r would contradict the leading-ideal condition. Conversely, a zero weak normal form gives $uf \in I$, and u is invertible. Using a generating set that is not a standard basis can give a nonzero remainder for an element of its ideal.

5.3 S-polynomials and Buchberger's criterion

For nonzero polynomials f, g , put $m = \text{lcm}(\text{LM}(f), \text{LM}(g))$. Their S-polynomial is

$$\text{spoly}(f, g) = \frac{m}{\text{LT}(f)}f - \frac{m}{\text{LT}(g)}g.$$

The leading terms cancel. For a global ordering, Buchberger's criterion says that a finite generating set G is a Gröbner basis precisely when every pair's S-polynomial reduces to zero by G . Buchberger's algorithm adds nonzero remainders of S-polynomials until this holds; termination follows from the ascending chain condition on monomial ideals. Local and mixed orderings require the corresponding standard-basis algorithms and weak normal forms.

II Modules

6 Modules

6.1 Modules and submodules

An **A-module** is an abelian group $(M, +)$ with scalar multiplication $A \times M \rightarrow M$ satisfying, for $a, b \in A$ and $m, n \in M$,

$$(a + b)m = am + bm, \quad a(m + n) = am + an, \quad (ab)m = a(bm), \quad 1m = m.$$

Examples include A itself, every ideal $I \subseteq A$, every quotient A/I , and vector spaces over a field K . Abelian groups are precisely $\mathbb{Z}$ -modules, with the usual repeated-addition action.

A nonempty subset $N \subseteq M$ is a **submodule** if it is closed under addition and scalar multiplication. It then contains zero and is closed under negatives, since $-n = (-1)n$. A module is **free** if it has a basis: every element has a unique expression as a finite linear combination of basis elements. The standard finite free module is A^r .

6.2 Homomorphisms, kernels, and cokernels

An A -linear map $\phi : M \rightarrow N$ satisfies $\phi(m + m') = \phi(m) + \phi(m')$ and $\phi(am) = a\phi(m)$. Its kernel is a submodule of M , and its image is a submodule of N . The set $\text{Hom}_A(M, N)$ is an A -module under pointwise addition and scalar multiplication. The dual module is $M^* = \text{Hom}_A(M, A)$.

For a submodule $P \subseteq M$, the **quotient module** M/P consists of cosets $m + P$, with addition and scalar multiplication induced from M . The quotient map sends m to $m + P$ and has kernel P . For a linear map $\phi : M \rightarrow N$, the **cokernel** is the module

$$\text{coker } \phi = N/\text{im } \phi.$$

Thus ϕ is injective exactly when $\ker \phi = 0$, and surjective exactly when $\text{coker } \phi = 0$. “Cokernel” denotes this quotient module, rather than the original map.

6.2.1 Isomorphism and correspondence theorems

- $M/\ker \phi \cong \text{im } \phi$.
- For submodules $P, Q \subseteq M$, $(P + Q)/Q \cong P/(P \cap Q)$.
- If $P \subseteq Q \subseteq M$, then $(M/P)/(Q/P) \cong M/Q$.
- Submodules of M/P correspond to submodules Q of M containing P , via $Q \mapsto Q/P$.

6.3 Operations and finiteness

The intersection of submodules is a submodule. For a family $(M_i)_{i \in \Lambda}$ of submodules of M ,

$$\sum_{i \in \Lambda} M_i = \left\{ \sum_{i \in F} m_i : F \subseteq \Lambda \text{ finite, } m_i \in M_i \right\}.$$

Only finitely many summands occur in each element. For an ideal $J \subseteq A$,

$$JM = \left\{ \sum_{i=1}^r a_i m_i : r \geq 0, a_i \in J, m_i \in M \right\}.$$

The module M is **finitely generated** if $M = Am_1 + \cdots + Am_n$ for some finite list of elements. For submodules $N, P \subseteq M$, the **colon ideal**

$$(N : P) = \{a \in A : aP \subseteq N\}$$

is an ideal of A , not in general a quotient module. In particular, $\text{Ann}_A(M) = (0 : M)$.

6.3.1 Torsion

An element $m \in M$ is **torsion** if $am = 0$ for some nonzero-divisor $a \in A$ (that is, multiplication by a on A is injective). These elements form the torsion submodule $\text{Tors}(M)$. The module is **torsion-free** if $\text{Tors}(M) = 0$, and a torsion module if $\text{Tors}(M) = M$. Over a domain every free module is torsion-free, but the converse fails. For example, the ideal $(x, y) \subseteq K[x, y]$ is torsion-free and is not a free $K[x, y]$ -module. A finitely generated torsion-free module over a principal ideal domain is free; the hypothesis on the base ring matters [4].

6.4 Presentations and Noetherian modules

A **finite presentation** of M is an exact sequence

$$A^m \xrightarrow{\Phi} A^n \longrightarrow M \longrightarrow 0,$$

where m, n are finite. Thus $M \cong \text{coker } \Phi$ and Φ is represented by an $n \times m$ matrix; its columns generate the relations among the chosen n generators of M . A matrix describing a map between free modules presents its *cokernel*, rather than its image automatically.

A module is **Noetherian** if every submodule is finitely generated. Equivalently, every ascending chain of submodules stabilizes, or every nonempty collection of submodules has a maximal element. If $N \subseteq M$, then M is Noetherian if and only if both N and M/N are Noetherian. Every finitely generated module over a Noetherian ring is Noetherian, and in this setting it is also finitely presented.

6.4.1 Nakayama's lemma and Krull intersection

Let $J(A)$ be the intersection of the maximal ideals of A . If $I \subseteq J(A)$, M is finitely generated, and $N \subseteq M$ satisfies $M = IM + N$, then $M = N$. In particular, $M = IM$ implies $M = 0$. This is a form of Nakayama's lemma [5].

If, in addition, A is Noetherian, then Krull's intersection theorem gives

$$\bigcap_{k \geq 0} I^k M = 0 \quad (I \subseteq J(A)).$$

One can obtain this using the Artin–Rees lemma and Nakayama's lemma; it is not an immediate consequence of Nakayama alone.

6.4.2 Minimal presentations over a local ring

Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $k = A/\mathfrak{m}$, and let M be finitely generated. A minimal generating set of M has $n = \dim_k(M/\mathfrak{m}M)$ elements. Choose a corresponding surjection $\pi : A^n \rightarrow M$, and write $L = \ker \pi$. A **minimal free presentation** is obtained by choosing $m = \dim_k(L/\mathfrak{m}L)$ minimal generators of L as the columns of Φ . Then $L \subseteq \mathfrak{m}A^n$, so all entries of Φ lie in $\mathfrak{m}$. Minimizing n alone does not ensure that the relation columns are minimal.

6.5 Localization of modules

For a multiplicatively closed $S \subseteq A$, the module $S^{-1}M$ consists of fractions m/s , with

$$m/s = m'/s' \iff t(s'm - sm') = 0 \text{ for some } t \in S.$$

It is an $S^{-1}A$ -module. An A -linear map $\phi : M \rightarrow N$ induces $\phi_S : S^{-1}M \rightarrow S^{-1}N$, $m/s \mapsto \phi(m)/s$. Localization is exact; consequently

$$\ker \phi_S \cong S^{-1}(\ker \phi), \quad \text{im } \phi_S \cong S^{-1}(\text{im } \phi), \quad \text{coker } \phi_S \cong S^{-1}(\text{coker } \phi).$$

For any A -module M , the following are equivalent: $M = 0$; $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$; and $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$. No finite-generation assumption is needed for this vanishing test.

7 Graded Rings and Modules

7.1 Graded rings and algebras

An $\mathbb{N}$ -graded ring is a ring with an additive direct-sum decomposition

$$A = \bigoplus_{d \geq 0} A_d, \quad A_d A_e \subseteq A_{d+e}, \quad 1 \in A_0.$$

Each element has a unique finite sum of homogeneous components; an element of A_d is **homogeneous of degree d** . The zero element may be regarded as homogeneous of any degree. Every ring has the trivial grading $A_0 = A$, $A_d = 0$ for $d > 0$.

A **graded K -algebra** is a graded ring that is a K -algebra with $K \subseteq A_0$ and each A_d a K -vector space. The additional condition $A_0 = K$ is often called *connectedness*; it is not part of the general definition of a graded K -algebra. For example, assigning positive integer weights w_i to the variables of $K[x_1, \dots, x_n]$ gives $\deg(x^\alpha) = \sum_i w_i \alpha_i$.

7.2 Graded modules and homogeneous submodules

A graded A -module is an additive direct sum

$$M = \bigoplus_{e \in \mathbb{Z}} M_e, \quad A_d M_e \subseteq M_{d+e}.$$

For $m = \sum_e m_e$, the element m_e is its homogeneous component of degree e . The sum is finite, though the indexing set is $\mathbb{Z}$. A submodule $N \subseteq M$ is **graded** or **homogeneous** if the following equivalent conditions hold:

1. $N = \bigoplus_e (N \cap M_e)$.
2. N is generated by homogeneous elements.
3. Whenever $m = \sum_e m_e \in N$, each m_e also belongs to N .

Under these conditions M/N inherits the grading $(M/N)_e = M_e / (N \cap M_e)$.

7.3 Homogeneous ideals and annihilators

A **homogeneous ideal** is an ideal homogeneous as a submodule of the graded module A . Its quotient ring inherits a grading. The annihilator of a graded module, $\text{Ann}_A(M) = \{a \in A : aM = 0\}$, is homogeneous: if $a = \sum_d a_d$ annihilates every homogeneous $m \in M_e$, the terms $a_d m$ have different degrees, so each must be zero. This holds for every homogeneous m , and hence each a_d annihilates M .

7.4 Homogeneous homomorphisms

An A -linear map $\phi : M \rightarrow N$ is **homogeneous of degree d** if $\phi(M_e) \subseteq N_{e+d}$ for every $e \in \mathbb{Z}$. The kernel, image, and cokernel of a homogeneous map inherit gradings. For a general linear map that is not homogeneous these submodules need not be graded. The standard category of graded modules uses homomorphisms of degree zero; grading shifts accommodate maps of other degrees.

8 Standard Bases for Modules

Let $R = K[x_1, \dots, x_n]_{>}$ and let $e_1, \dots, e_r$ be the standard basis of R^r . This is a free module, not a polynomial ring. Its module monomials are $x^\alpha e_i$, where $\alpha \in \mathbb{N}^n$ and $1 \leq i \leq r$; e_i has 1 in position i and 0 elsewhere.

8.1 Module orderings and leading submodules

A **module ordering** $>_m$ is a strict total ordering on these monomials such that

$$u >_m v \implies x^\gamma u >_m x^\gamma v, \quad x^\alpha > x^\beta \implies x^\alpha e_i >_m x^\beta e_i.$$

The ordering on each component agrees with the fixed ring ordering. For finitely many components it is a well-order exactly when the ring ordering is global. Common choices compare polynomial terms first (term over position) or component positions first (position over term).

If $f = cx^\alpha e_i + \text{smaller terms}$ with $c \neq 0$, set $\text{LM}(f) = x^\alpha e_i$, $\text{LC}(f) = c$, and $\text{LT}(f) = cx^\alpha e_i$. Divisibility is componentwise:

$$x^\alpha e_i \mid x^\beta e_j \iff i = j \text{ and } \alpha_k \leq \beta_k \text{ for all } k.$$

For $G \subseteq R^r$, its leading submodule is

$$L(G) = \sum_{g \in G \setminus \{0\}} K[x], \text{LM}(g) \subseteq K[x]^r.$$

Dickson's lemma applied separately to the r components shows that every monomial submodule is finitely generated.

8.2 Standard bases and normal forms

For a submodule $I \subseteq R^r$, a finite set $G \subseteq I \setminus \{0\}$ is a **standard basis** if $L(G) = L(I)$. For a global ordering it is a module Gröbner basis. Minimality with respect to leading monomials means that no $\text{LM}(g)$ belongs to $L(G \setminus \{g\})$. Full reduction also excludes reducible tail monomials. These are separate requirements.

The definitions of standard representations and normal forms from Chapter 5 apply to vectors. For a finite list $G = (g_1, \dots, g_s)$, a normal form $r = \text{NF}(f \mid G)$ satisfies $r = 0$ or $\text{LM}(r) \notin L(G)$, and

$$f - r = \sum_i a_i g_i, \quad \text{LM}(a_i g_i) \leq \text{LM}(f - r) \quad \text{when } a_i g_i \neq 0.$$

The zero representation is used if $f - r = 0$. For a weak normal form replace f by uf , with a unit $u \in R^\times$. The input f lies in R^r and G is a list of vectors in R^r ; the coefficients a_i lie in R . With a standard basis, the zero normal form detects submodule membership. In the global case a fully reduced remainder is unique.

8.3 Module S-polynomials

Let $f, g \in R^r \setminus \{0\}$ have leading monomials $\text{LM}(f) = x^\alpha e_i$ and $\text{LM}(g) = x^\beta e_j$. If $i = j$, let $\gamma_k = \max(\alpha_k, \beta_k)$ for each k , and set

$$\text{spoly}(f, g) = \frac{x^{\gamma-\alpha}}{\text{LC}(f)} f - \frac{x^{\gamma-\beta}}{\text{LC}(g)} g.$$

Both leading terms become $x^\gamma e_i$ and cancel. If $i \neq j$, no scalar monomial multiples can cancel the leading terms; such a pair is omitted from the S-pair computation (or assigned S-polynomial zero by convention). For global orderings Buchberger's criterion and algorithm extend to submodules with these S-pairs. Local and mixed cases again require weak normal forms and the corresponding standard-basis algorithms.

9 Computations in Singular

These examples use SINGULAR's exact polynomial arithmetic unless floating-point coefficients are explicitly requested. Every command ends with a semicolon. In polynomial input, x^2 abbreviates x^2 and xyz abbreviates xyz . Printed bases can differ by signs, nonzero scalar factors, or generator order; the ideal or submodule is the invariant. The listings were checked with SINGULAR 4.4.1. Command semantics follow the official manual [6].

9.1 Coefficient fields and polynomial rings

A ring declaration specifies the coefficient field, variables, and ordering.

```
ring QQx = 0, x, dp;           // Q[x]
ring F8x = (2^3, a), x, dp;     // F_8[x], field generator a
ring Cx = (complex, 30, I), x, dp; // approximate complex coefficients
ring Fp = 32003, (x, y, z), dp; // F_32003[x, y, z]
poly f = x2+y3+xyz;
f*f-f;
// y6+2xy4z+x2y2z2+2x2y3+2x3yz+x4-y3-xyz-x2
```

The characteristic-zero declaration uses $\mathbb{Q}$; it does not specify the real numbers. The finite field in the second line has eight elements. The complex declaration uses approximately 30 decimal digits of precision and is unsuitable for conclusions that require exact equality.

9.2 Monomial and matrix orderings

```
ring Lex = 0, (x, y, z), lp;
poly f = x2+xy+y2+z;
f; // x2+xy+y2+z
intmat W[3][3] = -1, -1, -1, 1, 0, 1, 0, 0, 1;
ring MatOrd = 0, (x, y, z), M(W);
poly f = imap(Lex, f);
f; // z+x2+xy+y2
```

The matrix has full rank. Its first row compares negative total degree, so smaller degree is larger and this is a local ordering. The remaining rows break ties; they put $x^2 > xy > y^2$. The polynomial is unchanged by `imap`; only its ordering and ambient ring change.

9.3 Quotient rings and equality

```
ring QuotBase = 0, (x, y, z), dp;
ideal I = x2+y2+5, z+x-yx3;
ideal G = std(I);
qring Quot = G;
ideal Zero = std(ideal(0));
reduce(x2+y2+5, Zero); // 0 in the quotient ring
reduce(z+x-yx3, Zero); // 0 in the quotient ring
```

This constructs $\mathbb{Q}[x, y, z]/I$ using a standard basis of the defining ideal. In the ambient polynomial ring, two polynomials f, g define the same coset if and only if `reduce(f-g, G)` is zero. Within the quotient ring, `reduce(f-g, Zero)` also reduces against its defining relations. One possible Gröbner basis of I ,

up to signs and order, consists of

$$\begin{aligned} &x^2 + y^2 + 5, \\ &xy^3 + 5xy + x + z, \\ &y^5 + 10y^3 + y^2 - xz + 25y + 5. \end{aligned}$$

9.4 Radicals and nilpotency

```
ring Rad = 0, (x,y,z), dp;
LIB "primdec.lib";
ideal P = x2+y2+xz-1;
ideal RP = radical(P);
reduce(RP, std(P));           // zero ideal
reduce(P, std(RP));           // zero ideal
```

The polynomial $x^2 + y^2 + xz - 1$ is square-free over $\mathbb{Q}$, so its principal ideal is radical. The two reductions verify equality of the computed radical and the original ideal, without depending on printed generators.

```
ideal I = xyz, x2, y2, x4+y4, y4+y7;
ideal J = std(I);
ideal RI = radical(I);
reduce(RI, std(ideal(x,y))); // zero ideal
reduce(ideal(x,y), std(RI)); // zero ideal
int k = 1;
while (reduce(y^k, J) != 0) { k++; }
k; // 2
int m = 1;
while (reduce(x^m, J) != 0) { m++; }
m; // 2
```

Here $I = (x^2, y^2, xyz)$ and $\sqrt{I} = (x, y)$. In A/I , the classes of x and y are nonzero but have square zero, so their nilpotency indices are both 2. Each loop increments its own counter and terminates because the corresponding class is nilpotent.

9.5 Krull dimension and membership

```
ring Dim = 0, (x,y,z), dp;
ideal I = xy+yz-1, yz+zx-1, xy+zx-1;
ideal J = std(I);
dim(J); // 0
reduce(y, J); // z
reduce(y-z, J); // 0
```

A Gröbner basis is $\{y - z, x - z, 2z^2 - 1\}$, up to order and scaling. Consequently

$$\mathbb{Q}[x, y, z]/I \cong \mathbb{Q}[z]/(2z^2 - 1).$$

This is a finite-dimensional $\mathbb{Q}$ -algebra (in fact a quadratic field), so its Krull dimension is 0. The command `dim(J)` computes the Krull dimension of the quotient, not a homological global dimension. The nonzero remainder of y shows $y \notin I$, whereas $y - z \in I$.

9.6 Normal forms

```
ring NFex = 0, (x, y, z), dp;
poly f = x2+y2+z2+xy+yz+zx+xyz-4;
ideal G = xy+y2-1, xyz+z2-3;
ideal H = std(G);
reduce(f, H); // x2+xz+yz
reduce(f-(x2+xz+yz), H); // 0
```

For this global ordering the fully reduced remainder is $x^2 + xz + yz$. Calling `reduce(f, G)` without first computing a standard basis can produce a different answer and is not a reliable ideal-membership test. An optional reduction flag can request leading-term-only reduction; such a result need not be fully reduced, so it must not be presented as the unique fully reduced normal form.

9.7 Submodules and matrix maps

A module specifies a submodule by its generating column vectors. The standard basis vector e_i is written `gen(i)`.

```
ring Mod = 0, (x, y, z), dp;
module M = [xy+1, z2+5, xyz], [y4, x2y2, z3y2];
matrix MM = M;
print(MM);
// xy+1, y4,
// z2+5, x2y2,
// xyz, y2z3
module N = MM; // recover its column generators
```

Thus M is a submodule of A^3 , for $A = \mathbb{Q}[x, y, z]$. The 3×2 matrix defines a map $A^2 \rightarrow A^3$ with image M ; as a presentation matrix it presents A^3/M .

9.7.1 Kernels and syzygies

```
matrix T[2][2] = x, x2y2, y2z4, xyz6;
module Ker = syz(T);
size(Ker); // 0: the zero submodule
```

The command `syz(T)` computes relations among the columns of T , that is, the kernel of the map $A^2 \rightarrow A^2$. Here $\det T = x^2 y z^4 (z^2 - y^3) \neq 0$. Since A is a domain, the map is injective: it becomes invertible over the fraction field, and A^2 embeds in that vector space. The nonconstant determinant does not make T invertible over A itself.

9.8 Sums, intersections, and colon ideals

Use a small example in A^2 to make the answers explicit.

```
ring Ops = 0, (x, y), dp;
module M = [x, 0], [0, y];
module N = [y, 0], [0, x];
module Sum = M+N;
module Cap = intersect(M, N);
ideal Col = quotient(M, N);
// Cap is generated by [xy, 0], [0, xy].
// Col is the ideal (xy).
reduce(Cap, std(module([xy, 0], [0, xy]))); // zero module
reduce(module([xy, 0], [0, xy]), std(Cap)); // zero module
reduce(Col, std(ideal(xy))); // zero ideal
reduce(ideal(xy), std(Col)); // zero ideal
```

The sum is $(x, y)e_1 + (x, y)e_2$, and the intersection is $(xy)e_1 \oplus (xy)e_2$, since $(x) \cap (y) = (xy)$. The result of $\text{quotient}(M, N)$ is the ideal $(M : N) = \{a \in A : aN \subseteq M\}$: the two component conditions require a to be divisible by both x and y . It is not the module M/N ; indeed N is not contained in M here.

9.9 Pruning a presentation

```
ring Pr = 0, (x, y, z), dp;
module M = [0, 1, xy+1], [y, xz, xz];
module P = prune(M);
print(P);
// y,
// -x2yz
```

Pruning removes a unit pivot and reduces the ambient free module while preserving the isomorphism class of the *cokernel*. In this example, subtracting $(xy + 1)$ times the second row from the third row, and then cancelling the unit pivot, gives

$$A^3/M \cong A^2/A(y, -x^2yz) \cong A/(y) \oplus A.$$

The last isomorphism follows by another invertible row operation. The routine does not assert that the original submodule is isomorphic to its pruned image, and it does not in general produce a minimal free resolution. For a non-global ordering, its permitted cancellations are more restricted; see the manual.

9.10 Weighted homogeneity

```
ring Gr = 0, (x, y, z), dp;
ideal I = y3-z2, x3-z;
qhweight(I); // 1, 2, 3
homog(I); // 0: not homogeneous in total degree
ring GrW = 0, (x, y, z), wp(1, 2, 3);
ideal IW = imap(Gr, I);
homog(IW); // 1
```

The weights satisfy $3w_y = 2w_z$ and $3w_x = w_z$, so the primitive positive choice is $(1, 2, 3)$; a positive scalar multiple gives the same homogeneous components up to renumbering. The generators have weighted degrees 6 and 3, respectively. The ideal $I = (y^3 - z^2, x^3 - z)$ is homogeneous for these weights,

and the quotient $\mathbb{Q}[x, y, z]/I$ inherits the grading. An ideal itself is not the quotient ring. The command `qhweight` may return a nonprimitive multiple; `homog(I)` returning zero means that the tested grading is *not* homogeneous.

9.11 Global and local standard bases

```
ring Glob = 0, (x, y), dp;
ideal I = x-x2, y;
ideal G = std(I);
reduce(x, G); // x: x is not in I globally
ring Loc = 0, (x, y), ds;
ideal IL = imap(Glob, I);
ideal GL = std(IL);
reduce(x, GL); // 0 in the local ring at the origin
```

Globally, $x(1-x)$ does not generate x . In the local ring $\mathbb{Q}[x, y]_{(x, y)}$, the element $1-x$ is a unit, so $IL = (x, y)$ and x belongs to it. The different answers reflect a change of ambient ring, not an inconsistency in the algorithms.

For a module example, a single generator is already a standard basis:

```
ring MG = 0, (x, y), dp;
module U = [x, y];
module UG = std(U);
vector v = [x2, xy];
reduce(v, UG); // zero vector
ring ML = 0, (x, y), ds;
module UL = imap(MG, U);
vector v = [x2, xy];
reduce(v, std(UL)); // zero vector
```

In both rings $v = x(x, y)$, so membership follows directly as well. Every multiple of a single nonzero vector has leading monomial divisible by that vector's leading monomial, which explains the standard-basis claim.

10 Conclusion

Standard bases connect ideal and module theory with effective computation. A correct computation requires a specified coefficient field, ambient ring, monomial or module ordering, and an appropriate standard basis. Global orders support terminating polynomial division and unique fully reduced remainders; local orders describe localized rings and require different reduction algorithms. Grading, localization, and presentations provide complementary ways to organize the resulting algebraic information.

Acknowledgements

I am grateful to Dr. Pratyusha Chattopadhyay for introducing me to commutative algebra and for her guidance and mentorship throughout the development of these notes. Her support and insight have deepened my understanding of the subject and shaped the work presented here.

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